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Simulating Urban Population Density with a Gravity-based Model

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Abstract—Theoretical justifications for the negative exponential urban density function were first proposed by urban economists, although some of their foundations have been criticized. From the geographer's perspective, the gravity-based model reported in this research uses a well-known concept (the "potential") to offer an alternative explanation. Using numerical analysis techniques, the model simulates various urban density patterns. By varying the model's parameters (the distance friction coefficient β and the city size), the numerical simulations do confirm two important empirical findings: the flattening of density gradients over time owing to transportation improvements, and flatter gradients in larger cities. The observed relationship between the β value and the urban density gradient, as established by this research, opens an avenue for empirical testing. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Since the classic study by Clark [8], there has been extensive work on urban population density in urban economics and geography from both theoretical and empirical standpoints. Among the various density functions estimated and tested in empirical studies (see McDonald [19] for a review), the negative exponential one has been the most widely used, with $D(r) = D_0 e^{-br}$, where $D(r)$ is the population density at distance r from the city center (CBD), D_0 is the density at distance zero, and b is the density gradient. Theoretical explanations for this function were first proposed by urban economists [20, 21, 23, 24] in a monocentric framework. The strength of these urban economic models is their grounding in microeconomic behavior.

Furthermore, since income and transportation cost are explicitly considered, these models elegantly explain the variations of the urban density gradient as a result of income growth and transportation improvements. However, economic theories and models are "simplification and abstractions that may prove too limiting and confining when it comes to understanding and modifying complex realities" [7, p. 527], especially when the spatial dimension is considered. First, the traditional monocentric assumption is questionable. In truth, more and more urban economists have recently abandoned this assumption, and developed multi-centric models. (See Ladd and Wheaton [16] for a brief survey.) Second, most of these models assume that the employment location pattern is pre-existing (i.e. exogenous), although a few models deal with the simultaneous location of employment and population [5, 27]. Third, the economic derivation of the negative exponential function requires a unit price elasticity for housing demand, which is not generally supported by empirical studies [18]‡. Two of the major findings of empirical studies on urban density are: (1) the flattening of gradients over time, and (2) flatter gradients in larger-size cities [19]. In the literature, the first observation has been explained by declining transportation costs and increasing income. However,

> "the fact that a larger urban population is associated with a flatter gradient requires further explanation. Mills [21], Edmonston [11], and Mills and Tan [22], for example, argue that a larger urban population implies a greater decentralization of employment and, hence, population. On the other hand, Muth [24, p. 154] argues that a greater supply elasticity of housing at the urban fringe is the cause of the observed result." [19, p. 380].

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‡This survey article lists the elasticity estimates of 13 separate studies, ranging from −0.36 to −0.87 (omitting the Chicago estimate of −1.31).

Taking the geographer's viewpoint, this paper uses a gravity-based model to simulate urban densities numerically, analyzes the resulting density patterns, and compares them with those derived from empirical analyzes of actual urban data. Gravity models (also known as "spatial interaction models") were first proposed because of their ability to replicate observed urban flow patterns, and thus were initially purely empirical (See [6] for a historical review). However, theoretical justifications for the gravity models were later proposed by (1) Wilson [30], using the entropy-maximizing principle (see also [10, 29]), and (2) Niedercorn and Bechdolt [25], using individual utility maximizing behavior (see also [1, 9, 15]). The second set of interpretations clearly support the view that gravity-based models are founded on particular theories of individual behavior, despite the opposite view primarily held by economists. It is thus these underlying theories that are tested in our research by assessing the fit between these predictions and observed data. The nature of gravity-based models allows for relaxing some of the restrictive assumptions of the traditional urban economic models (e.g., radial commuting towards the CBD) and, therefore, implies a wider applicability.

The paper is organized as follows. The second section presents the model, while the third section describes the spatial structure of a hypothetical urban area. The fourth section discusses the solution method, while the fifth section simulates various urban density patterns by varying model parameters, and analyzes the simulation results. Finally, the paper concludes with a brief summary and implications for further research.

THE MODEL

In physics, gravitation is a force that increases with mass and decreases with distance. Two types of models are based on this concept. The mainstream ones, referred to as *gravity models*, measure the interaction between two elements of space. "On the other hand, *potential models* aim at measuring the influence exerted by a set of masses on a unit of mass located at a given point in space" [4, p. 359]. The potential at point j is given by

$$V_j = b\sum_i \frac{x_i}{d_{ij}^{\beta}}, \tag{1}$$

where d_{ij} is the distance between the mass x_i and point j, β is the distance friction parameter, and b is a constant multiplier. In the socioeconomic context, x_i is usually represented by population size. Since potential models are derived from gravity models, they are also called *gravity-based models*.

Geographers and economists (e.g., [2, 3, 13, 14, 17, 28]) have used the potential or accessibility to all activities to measures urban agglomeration economies, and then determine population distribution. Consider an urban area composed of n equal-size cells. The population in cell j, x_j, may be expressed as a linear function of the potential there, with

$$x_j = c\left(b \sum_{i=1}^{n} \frac{x_i}{d_{ij}^{\beta}}\right), \tag{2}$$

where c is an unknown constant. Defining $k = 1/bc$, (2) is rewritten as

$$kx_j = \sum_{i=1}^{n} \frac{x_i}{d_{ij}^{\beta}}, \tag{3}$$

or, in matric notation:

$$k\mathbf{X} = \mathbf{AX}, \tag{4}$$

where $\mathbf{X}$ is a column vector of n elements $(x_1, x_2, \ldots, x_n)$, $\mathbf{A}$ is an $n \times n$ matrix with terms involving the d_{ij}'s and the distance friction parameter (β), and k is an unknown scalar. Clearly, eqns (3) or (4) is a system of n nonlinear equations with $(n + 1)$ unknowns (nx_i's and k). Since we are only interested in the spatial variation of population density, the population size in one of the cells can be normalized (e.g., $x_1 = 1$). Therefore, the system of equations can be solved.

The "potential" model of eqns (3) or (4) can be derived from the interactions between residential and employment locations within the context of a simplified Garin-Lowry model [14, 17]. Without differentiating basic and service employment, the employment vector $\mathbf{E}$ generated by a population vector $\mathbf{X}$ ("employment follows population") is $\mathbf{E} = \mathbf{BX}$, with a transaction matrix $\mathbf{B}$ based on a gravity model. Conversely, employment attracts population ("population follows employment") with $\mathbf{X} = \mathbf{CE}$, where the transaction matrix $\mathbf{C}$ is also based on a gravity model. Combining the two models yields $\mathbf{X} = (\mathbf{BC})\mathbf{X}$, i.e. $\mathbf{X} = \mathbf{AX}$, which is formally identical to eqn (4). If we assume that the elements of the bidirectional interaction matrix $\mathbf{A} = \mathbf{BC}$ can be modeled with the gravity formula, then the "potential" model of eqns (3) or (4) is fully explicated.

THE URBAN SPATIAL STRUCTURE

A generalized configuration of an urban area and its transportation network must be developed in order to generate the distance matrix. The urban area is here assumed to be partitioned into continuous circular rings, which are evenly divided into several sectors (see Fig. 1). The population at a given distance from the city center is assumed to be the same regardless of the direction from that center. The transportation network is not simply radial from the city center, as assumed in most urban models, but is a combination of radial (along any radius) and circular loops (along each ring). This approach better represents the real world and, more importantly, allows for interactions among various locations in different rings and within the same ring. All routes are assumed to be similar in terms of capacity, speed limit and road conditions.

The city is divided into n rings and m sectors, with $n \times m$ *equal-area* cells. As all cells have the same area, "population size" and "population density" in any cell are equivalent. A larger city has more rings than a smaller one. To make equal-area rings, the radii are as follows: $r_1 = 1$ and

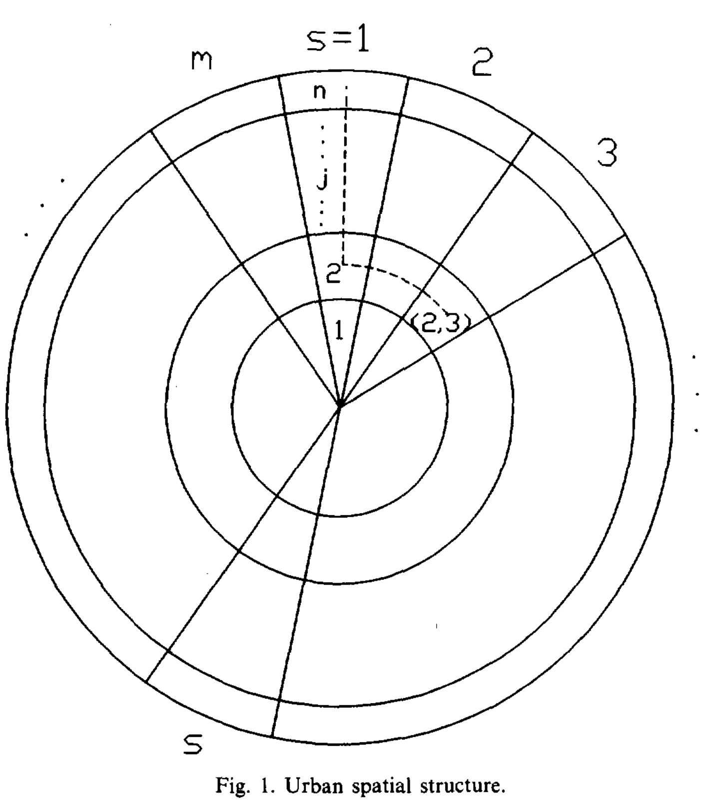

Fig. 1. Urban spatial structure.

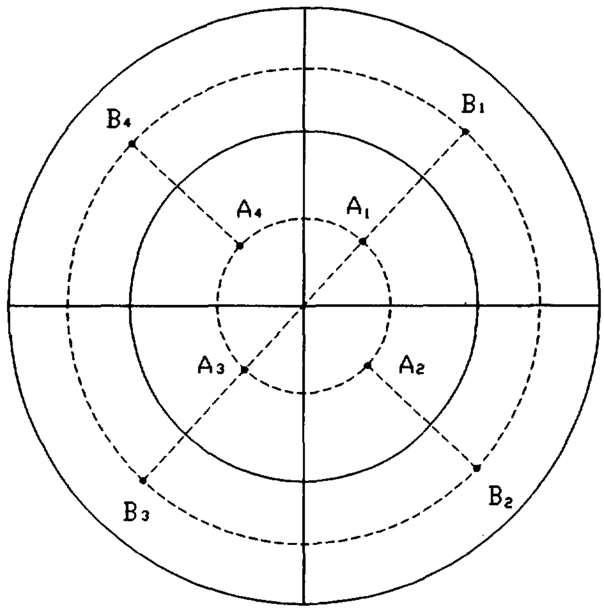

Fig. 2. Simplified urban spatial structure.

then, $r_2 = \sqrt{2}$, $r_3 = \sqrt{3}, \ldots, r_n = \sqrt{n}$. For example, the ith ring has an area equal to: $\pi(\sqrt{i})^2 - \pi(\sqrt{i-1})^2 = \pi$. Since the urban area is evenly divided into m sectors, each cell has an area π/m. The distance from the city center to the center point of the ith ring is thus:

$$T_i = \frac{r_i + r_{i-1}}{2} = \frac{\sqrt{i} + \sqrt{i-1}}{2}. \tag{5}$$

As all cells within a given ring have the same population, we need only to differentiate population size across rings. Cells in the first sector ($s = 1$) are taken as reference cells, indexed by j $(= 1, 2, \ldots, n)$, for the jth ring from the center. Cells elsewhere are indexed by (i, s), for the ith ring ($i = 1, 2, \ldots, n$) and the sth sector ($s = 1, 2, \ldots, m$) (see Fig. 1). Sectors are indexed clockwise from the reference cells. The distance between a reference cell (j) and any other cell (i, s), denoted by $d_{j(i,s)}$, is calculated following the shortest path rule. The algorithm for defining these distances is presented in the Appendix. For example, in Fig. 1, $d_{n(2,3)}$ is the distance between the nth reference cell and the cell in the second ring of the third sector, as indicated by the dashed line path.

The potential in a reference cell is related to all $m \times n$ cells, including itself. Within any ring, there are m identical cells, for $s = 1, 2, \ldots, m$. Using eqn (3), the model is specifically reformulated as

$$kx_j = \sum_{i=1}^{n} \left(\sum_{s=1}^{m} \frac{1}{d^{\beta}_{j(i,s)}} \right) x_i, \tag{6}$$

for all $j = 1, 2, \ldots, n$.

SOLUTION METHOD

We start with a simplified example in an effort to discuss the solution method. In Fig. 2, an urban area is divided into 2 rings ($n = 2$) and 4 sectors ($m = 4$), which make up 8 equal-size cells. Cells A_1 through A_4 have a population x_1 in each cell, while cells B_1 through B_4 have a population x_2. A_1 and B_1 are chosen as reference cells. Applying the shortest path rule to the reference cell A_1, the shortest distance from A_2 (A_4) is the arc A_2A_1 (A_4A_1); from A_3 (B_1, or B_3) it is the straight line A_3A_1 (B_1A_1, or B_3A_1); from B_2 (B_4) it is made up of two components, the line B_2A_2 (B_4A_4) and the arc A_2A_1 (A_4A_1). With regard to the reference cell B_1, the shortest distance from B_2 (B_4)

Table 1. Urban densities for various β values ($n = 10$)

Distance (r)	Density D(r)				
	$\beta = 2.0$	$\beta = 1.5$	$\beta = 1.0$	$\beta = 0.5$	$\beta = 0.2$
0.500	35.72	5.12	2.12	1.39	1.13
1.207	4.18	2.16	1.58	1.25	1.09
1.573	2.86	1.85	1.46	1.20	1.07
1.866	2.28	1.68	1.38	1.16	1.06
2.118	1.94	1.56	1.31	1.13	1.05
2.343	1.71	1.46	1.25	1.11	1.04
2.548	1.54	1.37	1.20	1.08	1.03
2.737	1.39	1.28	1.14	1.05	1.02
2.914	1.23	1.17	1.08	1.03	1.01
3.081	1.00	1.00	1.00	1.00	1.00

is the arc B_2B_1 (B_4B_1); from A_1 (A_3, or B_3) it is the straight line A_1B_1 (A_3B_1, or B_3B_1); from A_2 (A_4) it is made up of two segments, the arc A_2A_1 (A_4A_1) and the line A_1B_1. Distances within cells A_1 and B_1 are defined by rule (5) in the Appendix.

The radius of the inner circle is defined as 1, and $\beta = 2.0$. Using eqn (6), we obtain:

$$kx_1 = \left(\frac{1}{0.4463^2} + \frac{1}{0.7854^2} + \frac{1}{1.0000^2} + \frac{1}{0.7854^2}\right)x_1 + \left(\frac{1}{0.7071^2} + \frac{1}{1.4925^2} + \frac{1}{1.7071^2} + \frac{1}{1.4925^2}\right)x_2,$$

and

$$kx_2 = \left(\frac{1}{0.7071^2} + \frac{1}{1.4925^2} + \frac{1}{1.7071^2} + \frac{1}{1.4925^2}\right)x_1 + \left(\frac{1}{0.5776^2} + \frac{1}{1.8961^2} + \frac{1}{2.4142^2} + \frac{1}{1.8961^2}\right)x_2,$$

i.e.

$$kx_1 = 9.2617x_1 + 3.2410x_2,$$

and

$$kx_2 = 3.2410x_1 + 3.7254x_2.$$

Normalizing $x_2 = 1$, we obtain a system of two equations with two unknowns (x_1 and k), which can be further reduced to a quadratic equation with the variable k only. Solving for k yields: $k = 10.7558$, 2.2313. The second value of k ($=2.2313$) yields $x_1 = -0.4610$. As population size cannot be negative, only the value $k = 10.7558$ yields an acceptable solution of $x_1 = 2.1692$. This simple example illustrates a feasible solution approach. However, when n is large, this approach cannot be applied to solve eqn (3), which, basically, is a system of nonlinear equations. In the following, we propose an alternative, and more general approach†.

Considering the model's matrix notation in eqn (4), we see that if **X** is not a null vector, then k must be an eigenvalue of matrix **A**, and **X** the corresponding eigenvector. We use one of the most efficient and accurate numerical analysis methods to obtain all the eigenvalues and eigenvectors of a matrix‡. For instance, when $n = 10$, $m = 40$, $\beta = 2.0$, $x_{10} = 1$ (the population at the edge of the city is normalized), matrix **A** has 10 eigenvalues, all of which are real. By the Perron-Frobenius theorem, only the first (largest in absolute value) eigenvalue ($k = 299.7419$) yields a valid solution with all non-negative entries (the second column in Table 1). All other eigenvectors will have at least one negative value. We can see that the urban density decreases consistently from the center to the edge of the city.

†Newton's method can be used to solve systems of nonlinear equations, but requires an initial guess of the solution close enough to the true solution. In addition, Newton's method does not provide all feasible solutions.

‡We reduce matrix **A** to Hessenberg form by Gaussian elimination, then apply the QR algorithm to find the eigenvalues. Normalizing any one of the x_i's, and plugging any eigenvalue back into the system of equations, we obtain a linear system of n equations (one of them is trivial) with (n-1) variables. Selecting any (n-1) of the equations, we use the L-U factorization method to solve the linear system (see [26]). A FORTRAN program, that computes distances, generates matrix **A**, and determines the eigenvalues and the eigenvector **X**, is available from the authors upon request.

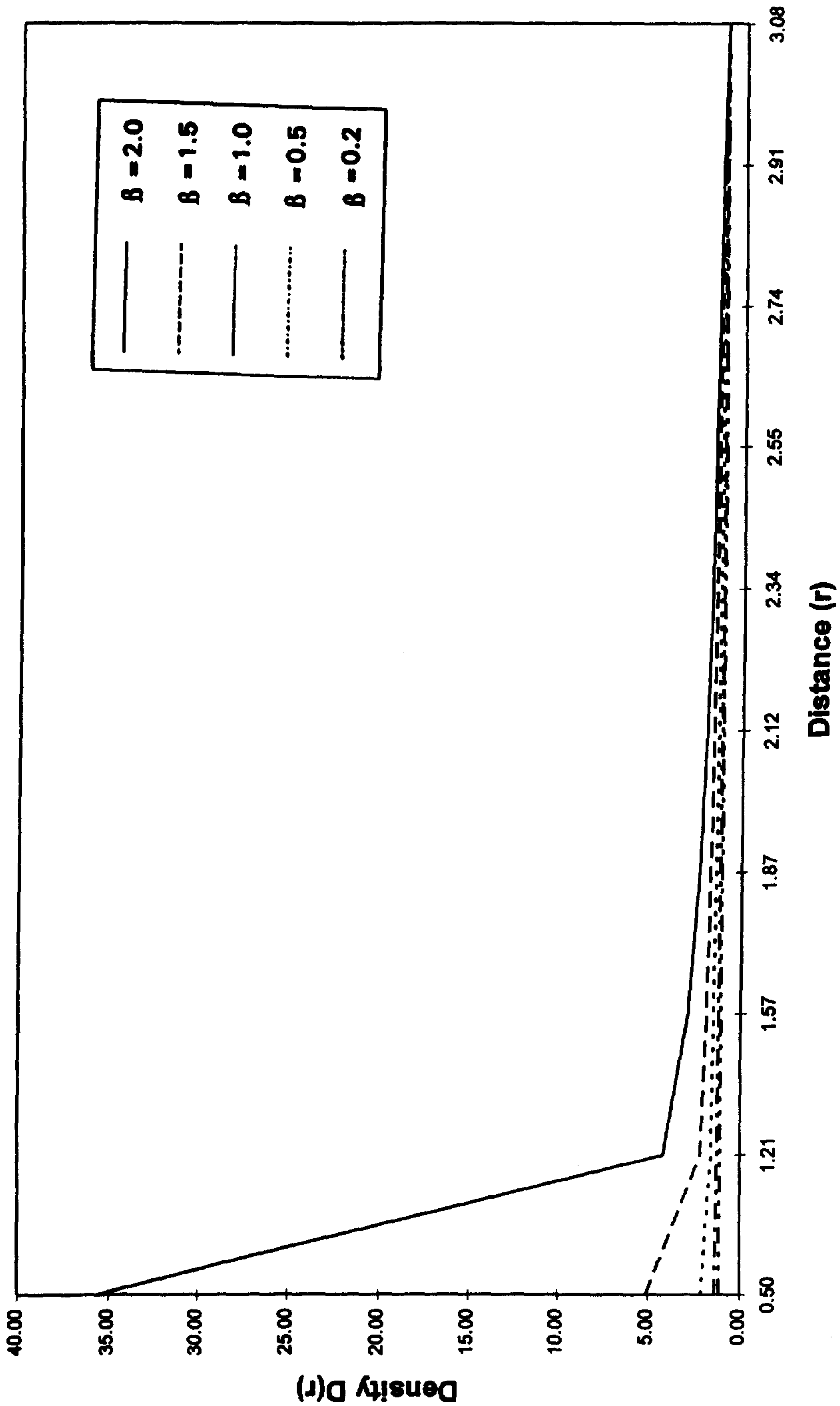

Fig. 3. Urban densities for various β values ($n = 10$).

Table 2. Urban densities for various city sizes ($\beta = 1.0$)

Ring (j)	Distance (r)	Density D(r) $n = 10$	$n = 15$	$n = 20$
1	0.500	2.12	2.13	2.21
2	1.207	1.58	1.65	1.69
3	1.573	1.46	1.55	1.59
4	1.866	1.38	1.48	1.53
5	2.118	1.31	1.42	1.48
6	2.343	1.25	1.38	1.44
7	2.548	1.20	1.34	1.41
8	2.737	1.14	1.30	1.37
9	2.914	1.08	1.26	1.35
10	3.081	1.00	1.22	1.32
11	3.240		1.19	1.29
12	3.390		1.15	1.27
13	3.535		1.11	1.24
14	3.674		1.07	1.22
15	3.807		1.00	1.19
16	3.936			1.16
17	4.062			1.13
18	4.183			1.10
19	4.301			1.06
20	4.416			1.00

SIMULATION OF URBAN DENSITY AND INTERPRETATION

In the model, the parameter β, indicating the degree of travel friction, plays a critical role. As the transportation technology improves over time, this friction becomes less important, and β decreases. We first simulate various urban density patterns by varying the value of β. The choice of β's variation range is based on Fotheringham [12]. Table 1 summarizes the results as β decreases from 2.0 to 0.2 (for the case $n = 10$). By plotting the results in Fig. 3, we clearly see that a declining β flattens the urban population distribution. We obtain the same results for the cases $n = 15$ and $n = 20$.

What then happens when city size increases? Since we always take the first radius (r_1) equal to 1, a larger number (n) of rings implies a larger urban area (with more population). Table 2 reports the results as n increases from 10 to 15 to 20 (for the case $\beta = 1.0$)†. Fig. 4, which is based on the results in Table 2, shows that a larger city has a flatter density pattern. Similar conclusions are obtained when β varies from 0.2 to 2.0.

Using regression analysis, we determine what density function best fits the simulated urban densities. Three functions (linear, exponential and log-linear) are tested. The results are reported in Table 3. When $\beta = 0.2$, 0.5, or 1.0, the exponential function has the largest $R^2(>0.97)$, while this is the case of the log-linear function for $\beta = 1.5$, 2.0. As expected, when β tends towards 0.2, the gradient (absolute value of b) becomes smaller and smaller, and the density curve becomes so flat that the exponential function converges onto the linear one.

In all city-size cases ($n = 10$, 15, and 20), when β becomes smaller, the gradient of the exponential function (absolute value of b) declines consistently, which indicates that improvements in transportation technology flatten the urban density curve‡. This supports the empirical observation that the gradients have flattened over time. Given the same β value, as city size increases from 10 to 15 to 20, we see a declining gradient in Table 3. This supports the observation that larger-sized cities have flatter gradients.

CONCLUSION

The gravity-based model of urban density is simply an extension of the well-known geographical concept of potential. The proposed spatial structure and transportation network for an urban area

†Based on the notion that the density at the edge of the city is close to the rural density, normalizing the density in the outer ring allows for the comparison across cities of different sizes.

‡Another possible cause is increasing income, which may reduce the relative cost of transportation, and thus the β value. In urban economic models, higher income stimulates more demand for housing, and makes the population distribution flatter. In the gravity-based model, the impact of increased income can be explained through its impact on the β value.

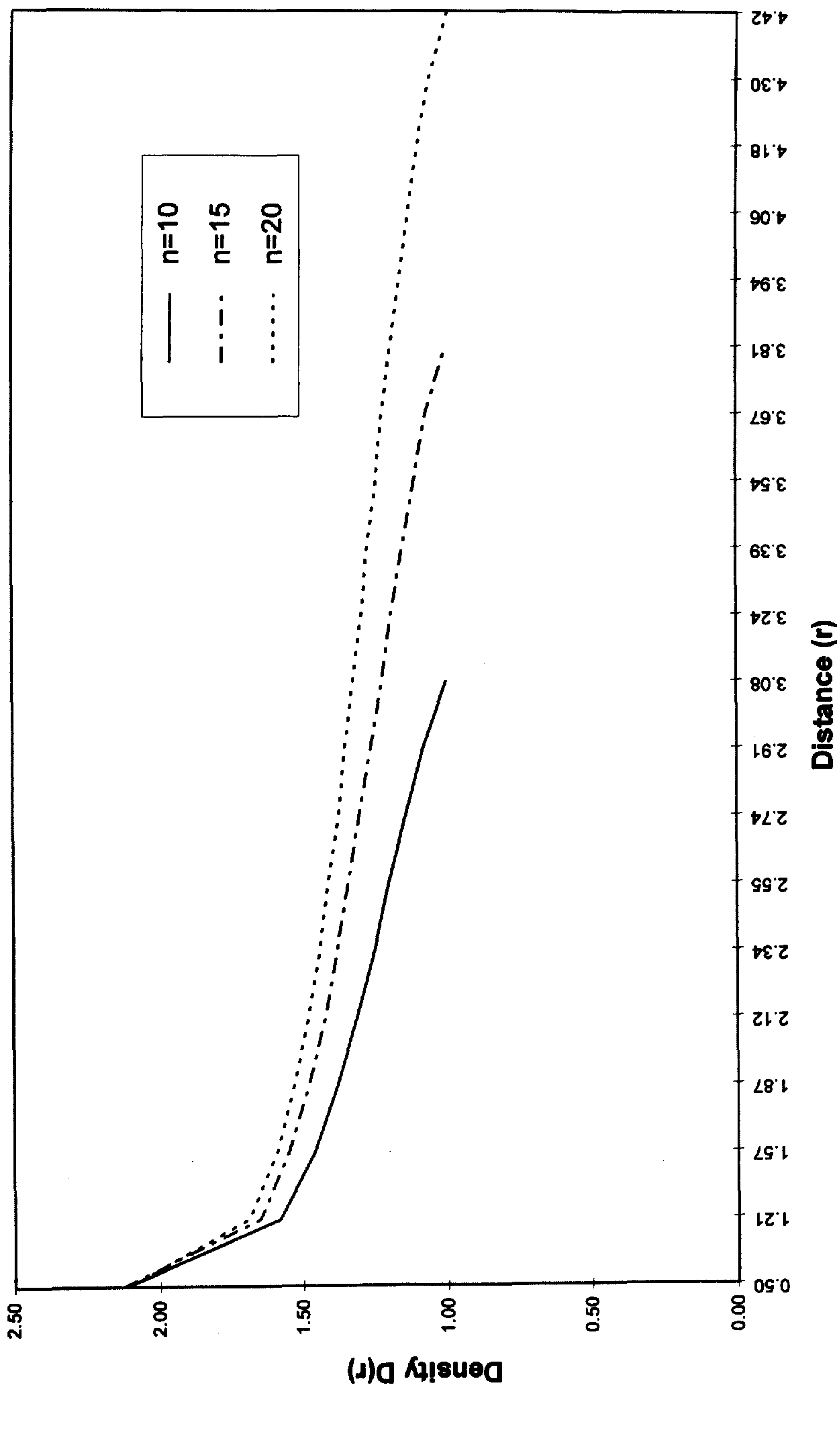

Fig. 4. Urban densities for various city sizes ($\beta = 1.0$).

Table 3. Various functions of simulated urban densities

City Size	β Value	Linear† a	Linear† b	Linear† R^2	Exponential‡ a	Exponential‡ b	Exponential‡ R^2	Log-linear§ a	Log-linear§ b	Log-linear§ R^2
	2.0	25.75	− 9.75	0.552	3.32	− 1.16	0.845	2.07	− 1.85	0.977
	1.5	4.50	− 1.26	0.742	1.61	− 0.53	0.903	1.03	− 0.82	0.982
n = 10	1.0	2.15	− 0.38	0.942	0.83	− 0.26	0.978	0.52	− 0.38	0.962
	0.5	1.43	− 0.14	0.990	0.38	− 0.12	0.995	0.24	− 0.17	0.943
	0.2	1.15	− 0.05	0.997	0.15	− 0.04	0.998	0.09	− 0.07	0.939
	2.0	20.90	− 6.42	0.435	2.83	− 0.78	0.763	2.14	− 1.57	0.948
	1.5	3.56	− 0.72	0.683	1.32	− 0.33	0.852	1.00	− 0.63	0.957
n = 15	1.0	2.07	− 0.28	0.941	0.78	− 0.20	0.974	0.57	− 0.35	0.973
	0.5	1.43	− 0.11	0.990	0.37	− 0.10	0.995	0.27	− 0.17	0.926
	0.2	1.15	− 0.04	0.997	0.15	− 0.04	0.998	0.10	− 0.07	0.924
	2.0	13.63	− 3.51	0.347	2.13	− 0.49	0.592	1.87	− 1.22	0.851
	1.5	2.89	− 0.43	0.621	1.07	− 0.21	0.772	0.31	− 0.48	0.909
n = 20	1.0	2.01	− 0.22	0.943	0.75	− 0.16	0.971	0.60	− 0.32	0.928
	0.5	1.43	− 0.10	0.991	0.37	− 0.08	0.994	2.89	− 0.16	0.916
	0.2	1.15	− 0.03	0.996	0.15	− 0.03	0.997	0.11	− 0.06	0.915

Note:
†Linear function: $D(r) = a + br$.
‡Log-transformation of exponential function: $lnD(r) = a + br$.
§Log-linear function: $lnD(r) = a + lnr$.

make it possible to extract some analytical results from a gravity-type model. Using numerical analysis techniques, the model was here solved by finding the eigenvalues of a symmetric matrix. Simulated urban densities tend to favor the negative exponential function. By varying the model's parameters (the distance friction coefficient β and the city size), the numerical results did indeed support two important empirical findings: flattening density gradients over time due to transportation technology improvements, and flatter gradients in larger cities. A direct connection between the β value and the urban density gradient is thus established by this research. If β were well-defined empirically for a given urban area, it would then be possible to test the proposed model by comparing the simulated results with actual density data. We strongly suggest others to consider such extensions of the work presented here.

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APPENDIX

Algorithm for Computing Distances

(1) When $j > i$, the distance between the reference cell j and cell (i, s) is composed of two segments (see the dashed line in Fig. 5), line EF and arc FG, with

$$d_{j(i,s)} = (T_j - T_i) + 2\pi T_i\left(\frac{s-1}{m}\right), \quad \text{(A1)}$$

where T_j and T_i are defined by eqn (5). This distance formula also applies to the case when $j = i$, except when $s = 1$ (see rule (5)).

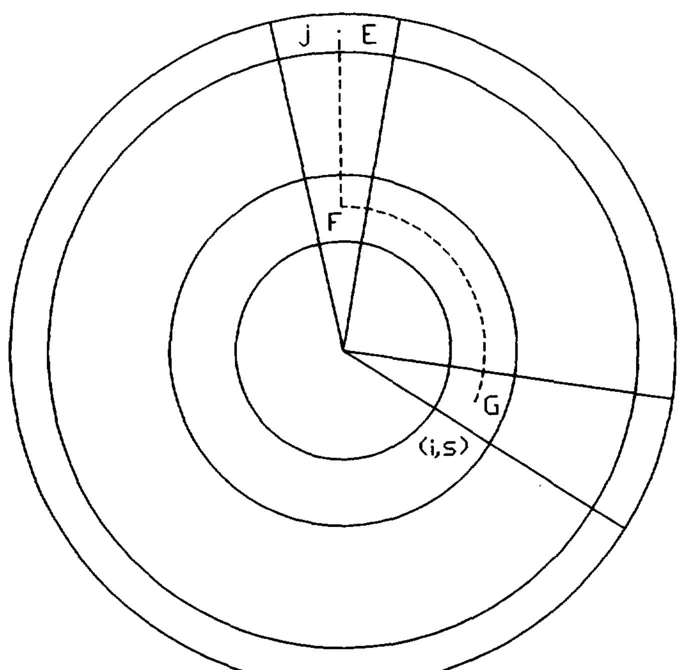

Fig. 5. Distance $d_{j(i,s)}$ when $j > i$.

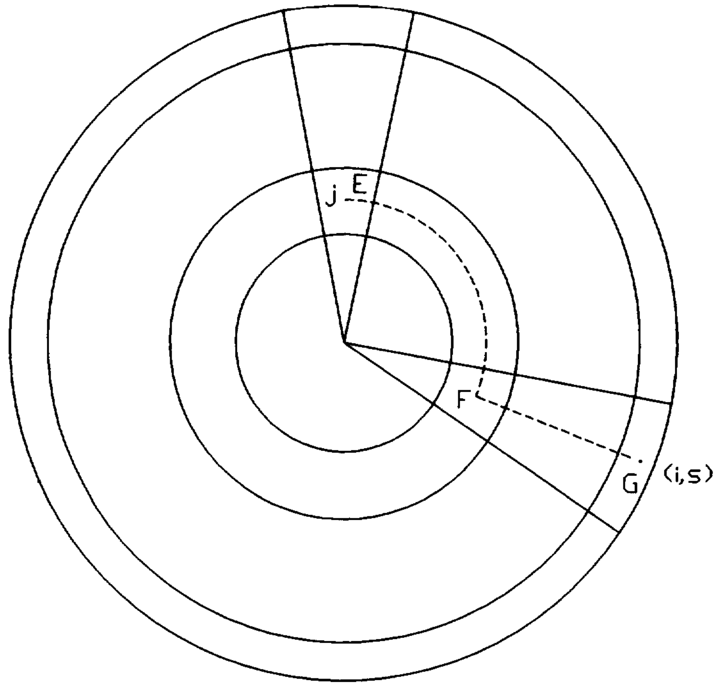

Fig. 6. Distance $d_{j(i,s)}$ when $j < i$.

(2) When $j < i$, $d_{j(i,s)}$ takes the inner (shorter) arc EF and line FG (see Fig. 6). Then $d_{j(i,s)}$ is equivalent to $d_{i(j,s)}$, and formula (A1) is rewritten as

$$d_{j(i,s)} = (T_i - T_j) + 2\pi T_j\left(\frac{s-1}{m}\right). \tag{A2}$$

(3) When $s > [m/2] + 1$ ("[ ]" operation means truncating real numbers to integers), cell (i, s) on the left-hand side is symmetric to cell $(i, m - s + 2)$ on the right side (see Fig. 7). Then $d_{j(i,s)}$

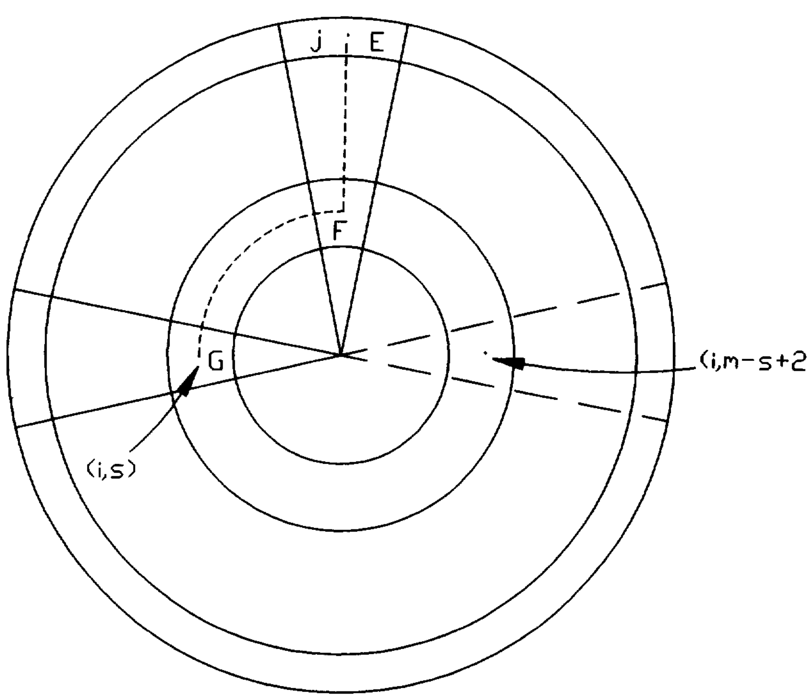

Fig. 7. Distance $d_{j(i,s)}$ when $s > [m/2] + 1$.